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ABSTRACT

This study was designed to follow up earlier work on mapping instruction. The two main goals were to: (1) test the effectiveness of mapping instruction as a general cure for "buggy" subtraction algorithms, and (2) explore two alternative explanations of how this new form of instruction works. It was hypothesized that mapping cures bugs either by imparting understanding, or through feedback and practice. About 80 children from the fourth, fifth, and sixth grades of an urban parochial school were given a written subtraction test designed to diagnose buggy algorithms. From this, eighteen students diagnosed as having buggy algorithms were chosen and divided into two groups approximately matched on race, sex, grade, and teacher rating of ability. After pretesting, one group received mapping instruction, the other received prohibition instruction. After training, both groups took both immediate and delayed posttests. Results indicated that mapping instruction is not always successful in curing bugs, and that the relationship between bugs and understanding is not straightforward. Also, some questions have been raised about what is learned in mapping instruction, as data suggest children can miss the general-trade-borrow analogy that is a major factor in the training. (MP)

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The Effects of Mapping Instruction on Children's Procedural Bugs in Subtraction

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Multidigit subtraction is one of the topics in elementary arithmetic that often causes difficulty for children. Some recent studies have shown that many children survive subtraction tests by inventing their own routines for borrowing that are systematic, but incorrect (Ashlock, 1976; Brown & Burton, 1978; Lankford, 1972; Young & O'Shea, 1981). These invented routines have been named "buggy algorithms" because they resemble computer programs with bugs in them (Brown & Burton, 1978). A striking feature of buggy algorithms is that they violate some basic principles of subtraction and borrowing, while producing borrowing notation that looks nearly correct (Resnick, 1982). Noting this feature of bugs, Resnick hypothesized that children who invent bugs do so without understanding the mathematical basis of borrowing. In other words, they view borrowing as a set of arbitrary, somewhat mysterious rules that one needs to use in order to solve problems on a test. From this hypothesis, Resnick reasoned that if children were led to understand borrowing, they would be able to correct their buggy algorithms. She developed a form of instruction called mapping instruction that aims to teach understanding of borrowing by setting up a detailed correspondence, or "mapping" between place-value blocks (illustrated in Figure 1) and written numbers. In a pilot study (Resnick, 1982) this instruction was successful not only in eliminating buggy algorithms but also in enabling children to explain why their former algorithms were incorrect and how the correct borrowing procedure works. These results were consistent with Resnick's hypothesis that mapping instruction teaches children to understand borrowing and that this understanding leads to correct performance. However, a second

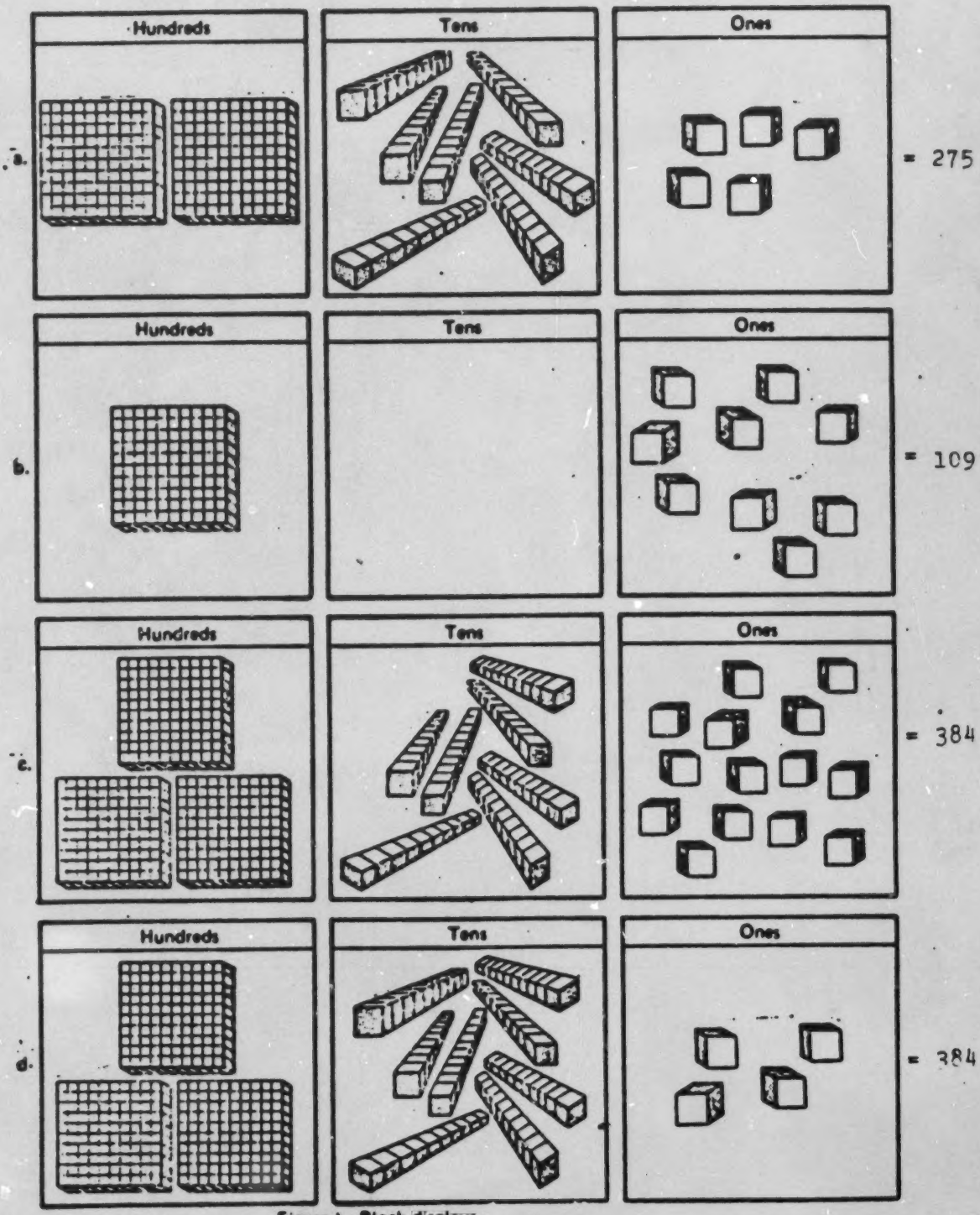


Figure 1 Block displays

possibility which Resnick considered is that children's bugs were cured simply through the direct feedback and practice embedded in the instruction. If so, then any understanding gained from the instruction was peripheral to correct performance.

The present study was designed to follow up Resnick's work on mapping instruction. This study had two main goals. The first was to test the effectiveness of mapping instruction as a general cure for buggy subtraction algorithms. The second was to explore the two alternative explanations of how the instruction works. The two explanations, again, are: 1) that mapping cures bugs by imparting understanding, and 2) that mapping cures bugs by feedback and practice. To do this we compared mapping instruction in its original form with a treatment called prohibition, which provided only the feedback and practice aspects of mapping instruction. We expected the prohibition treatment to correct some of the buggy algorithms already present, because of the step-by-step feedback available. However, we thought that if the effectiveness of mapping instruction as a cure for bugs is due primarily to teaching understanding, then the prohibition aspect alone should not be as effective. In particular, prohibition should not be sufficient to prevent new bugs from arising in later performance.

As background to the experiment, we first present an analysis of what we mean by understanding multidigit subtraction and how this understanding might influence written performance. Then we contrast the picture of full understanding with the type of understanding characteristic of children with buggy algorithms. Next, we show how mapping instruction can build upon children's prior knowledge to teach

the full understanding. And finally, we describe how mapping instruction might function as prohibition for some children.

Understanding Multidigit Subtraction

In this study, we are concerned with understanding an algorithm that is commonly taught in American schools. It can be summarized in the following set of rules:

1. Given a subtraction problem written in vertical form, subtract each bottom digit from the digit above it, starting with the rightmost column.
2. If the top digit is less than the bottom digit, borrow from the first non-zero top digit to the left of the current column.
3. When borrowing, decrement the digit from which you are borrowing by 1 and increment the digit to the immediate right by 10.

This algorithm would produce the "borrowing marks" shown in Figure 2 when applied to the problem 4059 - 382.

```
  3 9
 ~~4~~ 10 5 9
-  3  8  2
---------
  3  6  7  7
```

Figure 2. Example of borrowing algorithm referred to above.

As Resnick (1982) pointed out, one can know the full sequence of steps, or syntax of the algorithm with or without understanding the meaning, or semantics of the algorithm.

What does it mean to understand the semantics of the algorithm for written subtraction? Resnick, Greeno, and Rowland (1980) have answered this question in the form of a computer simulation of what one of the children in Resnick's study knew following mapping

instruction. In the Resnick et al. model, called MOLLY, understanding the semantics consists of a network of schemas which is used to explain the results of the borrowing procedure. In this network of schemas, one can identify four major components of understanding. These four components, which we will now describe, comprise the definition of understanding in the present study.

The first component of understanding is knowing the goals and subgoals which give structure to the algorithm. The goal structure can be stated as follows:

1. Top-level goal: Take away the whole bottom number from the whole top number.
2. Subgoal-1: Subtract within each column.
3. Subgoal-2: Whenever the top digit is less than the bottom digit in a column, borrow from another digit in the top number.

The second part of understanding is knowing the correct values of all the quantities involved in borrowing, including the value of each digit and the value of each increment and decrement. For example, in the problem shown above (Fig. 2), knowing the correct values would mean knowing that 1000 was borrowed from 4000, leaving 3000, that 900 was added to the hundreds column, and that 100 was added to the tens column.

The third part of understanding is recognizing that in borrowing there is a rule of compensation saying that the amount added to one column must be equal to the the amount taken from another column. For example, in the problem shown in Figure 3, the amount added to the tens column (100) is equal to the amount taken from the hundreds column (100 or 10 tens).

$$\begin{array}{r} {}^{4}\not{5}\,{}^{1}2\ 8 \\ -\ \ 6\ 2 \\ \hline 4\ 6\ 6 \end{array}$$

Figure 3.

Understanding the principle of compensation is dependent on knowing the values of the digits involved in borrowing. For example, if one thought that the little 1 in Fig. 3 was worth 10 and that the amount borrowed was worth 1, one would not be able to recognize the rule of compensation.

In addition to this general idea of compensation, the MOLLY model uses an exchange metaphor to explain how compensation is achieved when one always decrements by 1x and increments the adjacent digit by 10y. This metaphor says that borrowing is really an exchange of 1 of denomination x (e.g., 1 hundred) for 10 of the next lower denomination (e.g., 10 tens). Implicit in the exchange metaphor is the understanding that in our decimal number system, one of a given denomination is equivalent to ten of the next lower denomination. As we shall see later, the exchange metaphor is suggested to the student in the mapping instruction.

Another aspect of understanding compensation that is worth mentioning separately concerns borrowing across zero(s). This aspect is knowing that when borrowing across a zero (e.g., in Figure 4) the amount borrowed is distributed among two or more columns. For example, in the first problem, 100 is distributed between the tens and ones columns, as 90 + 10. In the second problem, 1000 is distributed among three columns as 900 + 90 + 10.

```
 5 9                2 9 9
 6 0 0              3 0 0 2
-1 8 2             - 5 4 3
-------            -------
```

Figure 4. Borrowing across zero(s).

Of course, the exchange metaphor can be used to further explain borrowing across zero in terms of multiple exchanges. For example, in 600-182 there are two exchanges: 100 for 10 tens and then 1 ten for 10 ones, and in 3002 - 543 there are three exchanges: 1 thousand for 10 hundreds, 1 hundred for 10 tens, and 1 ten for 10 ones.

Finally, the fourth part of understanding the semantics is recognizing that because of the compensation taking place in borrowing, borrowing preserves the value of the whole top number.

How would understanding the semantics of borrowing influence a student's written performance? There are at least two ways: 1) it can provide a basis for planning, and 2) it can be used to check the legality of moves.

In computational models of planning (e.g., Sacerdoti, 1977) two aspects of understanding are important: 1) knowledge of the goals of a procedure, and 2) knowledge of the constraints that must be honored in meeting the goals. The semantics of subtraction provide a goal structure for planning (viz., subtract the whole bottom number, subtract in columns, borrow when needed) and a constraint to be honored (viz., that the value of the top number be preserved, or more specifically, that any increment in one part of the number be balanced by an equivalent decrement in another part of the number). Simply adopting the goal structure for planning should prevent certain bugs, such as Smaller-From-Larger, which disregard one or more of the goals

or subgoals. For example, Smaller-From-Larger consists of subtracting the smaller digit from the larger digit in a column, no matter which is on top. When the top digit is subtracted from the bottom digit, as shown in Figure 5, the goal of subtracting the whole bottom number from the whole top number is abandoned, and the goal of borrowing is not considered.

```
 3 6 4
-1 8 7
-------
 2 2 3
```

Figure 5. Smaller-From-Larger.

Given that a student has adopted the goal structure for planning, honoring the rule of compensation will prevent many other bugs, such as Borrow-From-Zero and Borrow-Across-Zero, which involve incrementing without decrementing or decrementing too much (cf. Figures 6 and 7).

```
  5 6 12          4
 -1 0 8          5 0 12
 -------         -1 0 8
 4 1 4           -------
                 3 0 4
```

Figure 6. Borrow-From-Zero Figure 7. Borrow-Across-Zero

What if the student does not use the semantics in planning, but simply runs through all or parts of the subtraction algorithm automatically? It is still possible that his or her understanding of the semantics of borrowing can ensure correctness through the process of checking. In checking, the constraints (i.e., rule of compensation and preserving the value of the top number) would be most important. A student could check his or her borrowing notation to be sure that the values of the individual digits in the top number still added up to the initial value. For example, in Figure 7, such a check would

reveal that the new top number (400+12) was not equal to the old (500+2). Or a student could check individual borrowing operations, making sure that the increment was equivalent to the decrement. Looking at Figure 7 again, this check would reveal that the decrement (100) was not equal to the increment (10).

What do Children with Buggy Algorithms Understand about Multidigit Subtraction?

Resnick has noted that children who use buggy algorithms are performing as if they did not understand the semantics of subtraction. As Resnick has observed, the buggy algorithms identified by Brown and Burton (1978) disregard the semantics. For example, Smaller-From-Larger does not follow the goal structure, and Borrow-From-Zero violates the rule of compensation. Other evidence that children lack understanding of written subtraction is found in their answers to questions about the values of the little 1s and 9s written in borrowing. Resnick (1982) found that they read the face value of the digits, rather than their actual value. For example, given the problem in Figure 8, children often say that the 9 is worth just 9, rather than 90.

$$\begin{array}{r} {\scriptstyle 2\ 9} \\ 3\ \not{8}\ {\scriptstyle '}2 \\ -1\ 5\ 8 \\ \hline 1\ 4\ 4 \end{array}$$

Figure 8.

In addition, they often can not explain where the 9s come from in borrowing across zeros.

Does this mean that children with buggy algorithms do not understand the decimal number system at all? Probably not. Resnick (1982) found that when these children are given concrete models of place value, such as Dienes blocks, bundles of sticks, or colored chips, most of them can demonstrate understanding of value coding, and of 1-for-10 trading as a means of getting more of a denomination without changing the value of the whole number. Thus, it may be that children know the semantics of subtraction in some form but have not connected that knowledge to written borrowing.

Mapping Instruction as a Means of Teaching the Semantics of Written Subtraction

Mapping instruction was designed to build upon children's knowledge of concrete representations of place value. The particular form of mapping instruction discussed in this paper uses Dienes blocks, but other concrete representations, such as bundles of sticks, Montessori beads, or colored chips, could be substituted. Before discussing the hypothesized effects of the instruction, we will briefly describe the instruction itself.

In mapping instruction the student works out subtraction problems using both written numbers and Dienes blocks. First, the student is shown a subtraction problem written in vertical form, and is asked to represent the top number in blocks (cf. Figure 9 for an example). He or she is then given a large pile of assorted blocks that serves as the "bank" in making exchanges. The problem is then posed to the student as follows:

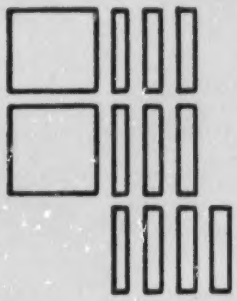
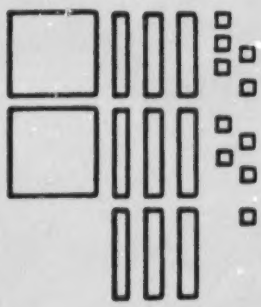
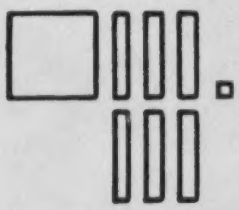
Problem: 300 – 139	Blocks Action or Writing Action
300 − 139	The child: 1. Displays larger number in blocks. 2. Writes problem in column-aligned format.
 2 3 0 0 − 139	3. Trades 1 hundred block for 10 tens blocks. 4. Notates the trade.
 2 9 3 Ø Ø − 139	5. Trades 1 ten block for 10 units blocks. 6. Notates the trade.
 2 9 3 Ø Ø − 139 / 161	7. In each denomination removes the number of blocks specified in the bottom number. 8. In each column notates the number remaining.

Figure 9. An outline of mapping instruction for borrowing.

1) Read the bottom number and try to take away that amount from the display of blocks. In Figure 9, for example, the goal is to take away 9 ones, 3 tens, and 1 hundred from the 3 hundred blocks.

2) Whenever you make a change in the blocks, record that change in the written numbers. The first change to be made in the example is to trade in 1 hundred for 10 tens. The trade is recorded on the written problem by decrementing the 3 in the hundreds column to 2, and incrementing the 0 in the tens column to 10. This recording is the "mapping" part of mapping instruction, as it shows how a particular change in the blocks corresponds to a change in the written numbers.

As Figure 9 illustrates, mapping instruction involves alternating between doing something in blocks and doing something in writing. Although the student eventually goes through the entire written subtraction algorithm, he does each written step only after he has done the corresponding step in blocks. We will now explain how these features of mapping instruction--viz., the fact that the student begins with the blocks, and the step-by-step alternation between the two representations--are important in teaching the semantics of written subtraction.

First, our insistence that the child start by trying to take away x in blocks before writing anything is important in teaching the goal structure outlined earlier (i.e., subtract the whole bottom number, subtract in columns, borrow if needed). The reason for this is that this goal structure is likely to be obvious in the context of subtraction in blocks, as we present it in mapping instruction. Because only the top number is represented in blocks and the child is asked to take away x ones and y tens from that display, the first two

goals are essentially given to the child. And, if there are only x ones in the display and the bottom number indicates that more than x ones must be taken away, the child can see that there are not enough ones in the display to do the subtraction, and that a subgoal should be to get more ones. Starting from the perspective of the blocks, then, confronts the child immediately with the goal structure that we want to teach. It also may make obvious the need for compensation, by allowing the child to experience conflict between getting more of a denomination and leaving the value of the blocks display unchanged. The hypothesis, then, is that starting with the blocks brings the semantics of subtraction to the child's attention.

A second hypothesis is that the step-by-step alternation between blocks and written numbers helps to link the blocks semantics to the written algorithm. When the child is asked to record the changes he or she has just made in blocks, he must find the right place in the written number and decide how to change the digits. The first few times he does it, the child might be guessing or might be shown what to do by the instructor. However, if the child is actually thinking about how to record a hundred-to-tens trade, for example, he will be trying to match the hundreds and tens blocks to the hundreds and tens columns, so that he can record that there are 10 more tens and 1 less hundred after trading. Mapping instruction thereby can give practice in associating block shapes to columns. And because the blocks represent the actual value of digits clearly, it is likely that students will also associate values with the digits they manipulate in borrowing. In addition, the alternation between blocks and writing can prompt the student to notice the analogy between the actions of

exchanging blocks and borrowing, and lead him or her to use the exchange metaphor discussed earlier.

Mapping Instruction as Prohibition

In spite of the fact that mapping instruction focuses so much on semantics, it is possible that its primary function for students may be to "tune" their algorithms for written borrowing through direct feedback and practice. In the mapping procedure, as designed in this study, the student is guided closely by the instructor and by the constraints of the blocks, so that any errors are corrected immediately. And, in the instruction, the student practices all the steps of the correct written algorithm, both within the blocks-to-writing mapping procedure and separately, without the blocks. Thus, embedded in mapping instruction is enough information for the following aspects of tuning to take place: 1) strengthening of the correct rules in the child's subtraction routine 2) weakening of any incorrect (buggy) rules, and 3) acquisition of any correct rules that were missing prior to instruction.

The Present Study

To review, the present study was designed to test the effectiveness of mapping instruction as a cure for subtraction bugs and to explore how the instruction works. Mapping instruction in its complete form was compared to prohibition instruction (to be described below, pp. 19-20), which provided only the feedback and practice aspects of mapping instruction. We expected mapping instruction to teach students to understand the semantics of written subtraction (which includes the goal structure, values of the borrow digits, rule of compensation, and rule of preserving the value of the top number)

and we expected this understanding to prevent the occurrence of bugs in post-instruction performance by providing a basis for accurate planning and/or checking. We hypothesized that prohibition instruction would correct some of the bugs, but would not prevent new bugs in future performance.

Method

Subjects

About 80 children from the fourth, fifth, and sixth grades of an urban parochial school were given a written subtraction test designed by Brown and Burton (1978) to diagnose buggy algorithms. From this group we chose 20 students who were diagnosed as having buggy algorithms to be in the study. Two of the students were dropped from the study after our screening interview because they seemed to lack some prerequisites for our instruction (e.g., the ability to count by hundreds).

The 18 students who comprised the final sample were divided into two groups that were approximately matched on race, sex, grade, and teacher rating of ability. The exact composition of the two groups is presented in Table 1.

Procedure

Overview. Children in both groups were given two preliminary interviews (screening and pre-training) to establish that they had the prerequisite knowledge for our instruction. Then, they were given a more structured interview that served as a pretest (and later as a posttest) for the instruction. After the pretest, one group received mapping instruction, while the other received prohibition instruction. Both groups were then given an immediate posttest (2-7 days after

Table 1

Composition of Groups along various Dimensions

Dimension	Group	
	Mapping	Prohibition
Ability[a]		
Average-High	7	7
Low	2	2
Sex		
Male	4	4
Female	5	5
Grade		
4	4	3
5	4	4
6	1	2
Race		
Black	8	7
White	1	2

[a] Based on teacher ratings.

instruction) and a delayed posttest (about 4 weeks after the immediate posttest). Both posttests were essentially the same as the pretest, except that different numbers were substituted in the examples. (Table 2 summarizes the steps of the study.)

Each interview was administered individually. The screening, pre-training, and instruction interviews were conducted by one pair of experimenters (one interviewer and one notetaker) and the pre- and posttest interviews were conducted by another pair, who did not know which type of instruction the children received. Each interview was audiotaped and lasted approximately 40 minutes, except for the screening, which lasted about 20 minutes. We gave the screening interview to most of the children within the first two weeks of the study. Thereafter, we worked with two or three children in parallel until they completed the instruction and immediate posttest. For most students, about one and one-half weeks elapsed between the pre-training and the immediate posttest. Because of scheduling constraints and student absences, there was a considerable difference among students in when they received the delayed posttest. The possible effects of variation in time of test will be discussed as part of the results.

Screening. This session served three functions: to test the children's knowledge of counting and place value, to introduce them to Dienes blocks if they were not already familiar with them, and to test the stability of their subtraction bugs. It included both an interview and a written subtraction test. The first two sets of questions asked students to read numerals and to count by ones, tens, and hundreds. The remaining questions assessed their ability to

Table 2

Summary of the Procedure

1. Screening
2. Pre-training
3. Pretest
4. Instruction (Mapping of Prohibition)
5. Immediate posttest
6. Delayed posttest

represent written numbers with the Dienes blocks and to "read" given blocks displays--i.e., tell what number a display represented.

Pre-Training. One purpose of this training was to ensure that all of the students could represent and solve subtraction problems with the blocks, using the one-for-ten trading procedure. Another purpose was to continue to assess students' pre-instructional knowledge. Thus, we used a quasi-discovery approach to training. In the beginning of each section of the training we gave students an opportunity to work out the problems, or to describe how they would work them out. If a child attempted a wrong solution or failed to respond to the question, we tried several layers of prompts, and if the prompts failed, we demonstrated the correct procedure for the child. After the child did one problem correctly, he or she was told that that method was the one we wanted him or her to use. Then we gave the child several variations of the problem for practice until he or she could do all types correctly, at least twice consecutively.

The first set of problems included variations of the following: given a canonical display of blocks (i.e., one with no more than 9 blocks in any denomination), add more blocks to a certain denomination, but keep the total value of the display the same. For example, the first problem was to show 34 (initially represented as 3 tens and 4 ones) using more than 4 ones blocks. In these problems we wanted students to develop a procedure for trading that they could apply later within subtraction problems.

The second set of problems were subtraction problems to be worked out in blocks only. The child was told to set up the top number (minuend) in blocks and to try to take away the number of ones, tens, and hundreds shown by the bottom number (subtrahend), starting with the ones. The first problem required no trading, the second required trading a ten for ten ones, and the third, a hundred for 10 tens. Later problems called for a double trade--i.e., trading a hundred for 10 tens, and then a ten for 10 ones.

The final section of the pre-training focused on what Resnick (1982) has called the "result map." We say that a student has a result map if he or she recognizes that the blocks procedure and the written procedure produce the same result for any given problem. In the result-map exercises, we asked students to do a simple problem in writing and then predict what answer they would get if they did the same problem in blocks. Then we asked them to do the problem in blocks. If they did not predict correctly, we gave them a second problem and repeated the question. Once they seemed to understand the idea of result mapping with simple problems, we gave each student a problem chosen to elicit his or her bug. The student did the problem in writing first, then in blocks. The purpose of the exercise was to give each student an opportunity to correct his or her written bug after comparing the written result to the blocks result. The experimenter did not press the student to do so, she only asked which answer was right and whether the student could figure out what went wrong.

Pretest. A form of this test was given after pre-training and then twice following instruction, as the immediate and delayed posttests. The test served two functions: 1) a further check on whether the students had the prerequisites for instruction, and 2) as a measure of learning from our instruction. A copy of the test in the Appendix shows how the items correspond to the knowledge we wanted to assess. Section A is concerned with pre-requisites for mapping instruction, and includes: 1) knowing the value of individual digits in a regular number (i.e., a number without borrowing notation) 2) trading, 3)result mapping, and 4) subtracting with blocks. Section B is concerned with the semantics of written subtraction and probes knowledge of: 1) the top-level goal to subtract the whole bottom number from the whole top number, 2) the value of digits in borrowing notation, and 3) the rule of compensation (the decrement must equal the increment), including whether or not the child uses the exchange metaphor to explain compensation. We did not specifically test the idea that the whole top number is preserved. Also, because of time constraints, we did not probe the students' understanding of the semantics in all types of subtraction problems, nor did we ask for both verbal responses and demonstrations with blocks in all problems. Table 2 in the Appendix shows the range of questions we did ask about values of digits involved in borrowing. Finally, Section C of the pretest assesses students' knowledge of the syntax of borrowing. It includes a test in error checking and a written diagnostic test, like the one given in the screening interview.

Mapping Instruction. The mapping instruction was structured much like the pre-training. The problems were presented from easiest to most difficult, and in such a way that students were allowed to figure out the correct mapping procedure themselves, if they could do so quickly. If a student seemed lost, the experimenter used prompts and then demonstrations, if necessary, to help him or her do the mapping correctly. The mapping procedure was taught as a process of recording the blocks moves. The experimenter asked the child to start doing a subtraction problem with the blocks, and then stop after each step and record the changes on the written numbers. We expected the children to approach mapping as an extension of blocks subtraction, which they knew how to do. However, we thought that a number of children might not be able to record the blocks changes correctly, either because they were not familiar enough with standard borrowing notation or because they did not think it was possible to show more than 9 in a column (a reasonable generalization from the carrying rules in addition). Our prompts were designed to help these children find a way to record blocks changes by noting the new number of blocks in a denomination and trying to make the digit in the corresponding column match that number. After they succeeded in recording the results of a trade and a take-away operation, we expected the students who already knew a partially correct written subtraction algorithm (i.e., most students with bugs other than Smaller-From-Larger) to recognize that the marks they made to record blocks moves were the same as those in their normal borrowing algorithm. (The experimenter did not point out this correspondence.)

If the student did not know anything about borrowing before

instruction, we hypothesized that he or she would build up a new written subtraction algorithm through practice during the instruction.

The mapping instruction consisted of three parts: 1) learning and practicing the full blocks-recording procedure with different types of problems, 2) "fading," or practicing the recording procedure while the experimenter pretended to move the blocks, and 3) doing the written procedure alone. An important part of the fading was that the child had to give the experimenter verbal instructions about what to do with the blocks.

Prohibition instruction. This instruction consisted of two parts: 1) more training in blocks subtraction, as in pre-training, and 2) prohibition instruction. The blocks training was included to make the prohibition session more like the mapping session, by making it longer and giving the children more exposure to the blocks. Prohibition consisted of three phases: 1) learning and practicing the specified procedure with different types of problems, 2) fading, and 3) doing the written procedure alone.

The experimenter (E) began the prohibition instruction by introducing herself as the student's subtraction robot, who would do problems for the student (S), but needed explicit directions about what to write. E and S worked through a set of five problems--the same five that were used in mapping instruction--with S telling E what to do. If S told E to do anything wrong or in the wrong order, E said, "I am not programmed to do it that way. Try again." Otherwise, E wrote what S told her to write. If S could not tell E the correct move after a few guesses, E wrote the correct move and then asked S to continue from there. After doing the first five problems, E and S

worked through a list of about 20 more problems, each of which could elicit certain bugs. (The correspondence between types of problems and types of bugs is detailed in Table 3.)

In "fading," the student took over the writing, but still had to tell E what each step would be before he actually wrote it. Finally, S was asked to work out some problems in writing silently.

Scoring

The written subtraction tests were checked and then analyzed by me and by Brown and Burton's diagnostic system, DEBUGGY, to determine the types of bugs exhibited. The diagnosis included the name of the bug and a measure of its systematicity (a ratio of the number of times it appeared to the number of times it could have appeared; for example, if there were six problems requiring borrowing over a zero (e.g., 302-179) and the student used the bug Don't-Decrement-Zero four times, the ratio would be 4/6).

In addition to diagnosing the bugs with Brown and VanLehn's system, we classified each different bug as being one of four types, based on two criteria: 1) how many different types of problems on the diagnostic test would elicit the bug, and 2) how much of the correct "core procedure" for multidigit subtraction is missing or distorted. Table 3 presents the four classes of bugs. Class E bugs are those which occur in all problems requiring borrowing. Each bug in this group either avoids borrowing altogether (e.g., Smaller-From-Larger) or does the incrementing and/or decrementing incorrectly (e.g., Borrow-From-Bottom, Borrow-Unit-Diff). We could say that these bugs are the most serious, because they reflect little knowledge of borrowing and produce a high percentage of wrong answers on tests.

Table 3

Classes of Bugs and the Types of Subtraction Problems in which They Occur

---

Class E Bugs: Occur in any problem with one or more borrow.

Examples *

- add borrow carry subtract
- add borrow decrement
- add borrow decrement without carry
- blank instead of borrow
- borrow add is ten
- borrow from bottom
- borrow from larger
- borrow no decrement
- borrow unit difference
- doesn't borrow
- smaller from larger
- subtract top from bottom
- subtract copy least bottom most top
- decrement on first borrow

Class D Bugs: Occur in problems with two or more consecutive borrows.

Examples *

- borrow once then smaller from larger
- borrow once without recurse
- borrow only once
- borrowed from don't borrow

---

* Names of specific bugs listed in Brown and VanLehn (1980).

Table 3 (continued)

Class C Bugs: Occur in problems with borrowing across zero.

Examples *
- borrow across 0
- borrow add decrement instead of zero
- borrow decrementing to by extras
- borrow from bottom instead of zero
- don't decrement zero until bottom blank
- don't decrement zero
- smaller from larger instead of borrow from zero
- stops borrow at zero
- treat zero as nothing
- zero instead of borrow from zero
- borrow from zero
- borrow from zero is ten
- borrow from zero and left OK
- borrow diff 0-n=n and small-large = 0
- borrow won't recurse

Class B Bugs: Occur in problems with special features or patterns.

Features/Patterns	Examples of Bugs *
1. blanks	stutter subtract
	simple problem stutter subtract
2. zero over a blank	ignore zero over blank
3. N-N in a column	diff N-N=N
	N-N=9 plus decrement
4. N-0 in a column	diff N-0=N
5. Borrow in ones but not in tens	once borrow, always borrow
	always borrow left
6. Borrow for a 1	borrow into 1 = 10
	diff 1 - N = 1
7. Borrow for a 0	borrow 10 plus next digit into 0
	diff 0-N=0
	diff 0-N=N
	treat top 0 as 9
	treat top 0 as 10

---

* Names of specific bugs listed in Brown and VanLehn (1980).

Table 3 (continued)

Features/Patterns	Examples of Bugs
8. Borrow over a blank	forget borrow over blanks quit when bottom blank subtract 1 over blanks
9. Borrow from a column where top = bottom	borrow skip equal N-N=1 after borrow
10. Borrow from a column where top < bottom	borrow don't decr. unless bot. smaller borrow across top smaller decr. to borrow don't decr. top smaller
11. Borrow from a column where top > bottom	borrow only from top smaller
12. Borrow from 1	0-N=0 after borrow 0-N=N after borrow borrow from 1 is 9 borrow from 1 is 10 borrow treat 1 as 0 decrement 1 to 11 double decrement 1
13. Borrow from 1 over 1	1-1=0 after borrow 1-1=1 after borrow
14. Borrow across zero over a blank	borrow across 0 over blank borrow ignore 0 over blank don't decrement 0 over blank increment 0 over blank
15. Borrow across zero over zero	borrow across 0 over 0 don't decrement 0 over 0
16. Borrow across zero for a zero	decrement on all multiple 0 difference 0-N=N when borrow from 0 don't decrment second zero
17. Borrow across 2 zeros	borrow from all 0 decrement leftmost zero only decrement multiple zeros by no. to right stops borrow at multiple zero

Class D bugs are those which occur in problems requiring two or more consecutive borrows (e.g., 724-596). In these bugs, the core borrowing procedure is intact, but for some reason, it is not recursive.

Class C bugs occur in problems requiring borrowing over a zero (e.g., 305-139). Class C bugs have the core borrowing procedure intact, but they are missing the procedure for borrowing over a zero. Each bug in this class uses a different strategy for handling this impasse.

Finally, Class B bugs are bugs which seem to be triggered by special features of problems (i.e., by "interrupt conditions" in Brown & VanLehn, 1980). One could argue that they are less serious than the other bugs, because they occur less frequently on a typical test (e.g., 0-N=0 only occurs on problems with 0-N in a column), and, more importantly, because if a student has only a Class B bug, and can do problems with two borrows and with zeros--as long as the special pattern is not present--then we could say that he or she knows the core procedure for borrowing, and simply has to generalize it to the special patterns.

For the screening interview, a simple checklist was used, with each task listed (e.g., counting from 30 to 120 by tens). A student's performance on a task was coded "0" if correct on the first attempt and n if correct on the nth attempt (noting whether or not the student was prompted by the experimenter).

The pretest and posttest items were scored on a pass-fail basis. The copy of the test in the Appendix shows examples of correct responses to the items. (Variations of the responses listed were also accepted.) Five scores were determined for each individual. Table 4 shows the 5 scores that were determined for each student.

The pre-training and instruction interviews were not scored quantitatively. As with the pre- and posttests, full protocols were obtained from transcriptions of the audiotapes together with an observer's notes. The analysis of the instruction protocols will be discussed in the results and discussion section.

Results and Discussion

Pretest Results

The pretest assessed three main areas of knowledge: 1) performance of written multidigit subtraction, 2) knowledge we considered to be prerequisite to learning from mapping instruction, and 3) knowledge of the semantics of written subtraction. The mapping and prohibition groups did not differ reliably in any of these areas. Both groups showed mastery of the prerequisites, which included items on place value of regular digits, trading, subtracting in blocks, and recognizing a "result map" between blocks and written subtraction ($\bar{x}$=90.4% correct for mapping group, $\bar{x}$=92.1% correct for prohibition group). Each student in both groups had at least one bug in his or her written subtraction algorithm. Table 5 shows the distribution of types of bugs in the two groups. Turning to the semantics of borrowing, the prohibition group did somewhat better than the mapping group on items asking about the values of borrow digits but the difference was not reliable (cf. Figure 10). Only two students in

Table 4

Explanation of Scoring for Pretest

Score 1 = Percent correct of responses to questions on PREREQUISITE KNOWLEDGE.

These questions are in Section A of the Pretest in Appendix D (pp. 1-4) under 4 subheadings:

1. Value of regular digits and block denominations
2. Trading
3. Result mapping
4. Blocks subtraction procedure

Each specific response that was scored is underlined and numbered (starting with R1).

Score 2 = Pass/fail for question on GOAL STRUCTURE.

This question is number 5 in Section B (pp. 4-5) of the Pretest.

A Pass was given if the student said that you can't do the problem because the whole bottom number is bigger than the whole top number. (R1)

A Fail was given if the student said either
1) that it was all right to subtract in the first two columns, but not in the left column, or
2) that you could do the problem by skipping the left column, writing 0 there, or borrowing from the tens.

Score 3 = Percent correct on questions asking for VALUES OF BORROW DIGITS.

These questions are in Section B of the pretest (pp. 5-9) under 2 subheadings:

6. Verbal expression of values
7. Block expression of values

There was a total of 21 responses scored in these sections (R15 - R35).

Table 4 (continued)

Score 4 =	Pass/fail for honoring the RULE OF COMPENSATION in answering questions about the value of borrowed quantities. Pass was given if the following responses were correct: R19, R20, and R35. Fail was given if any of the above responses was incorrect.
Score 5 =	Pass/fail for mentioning the EXCHANGE METAPHOR in explaining borrowing. Pass was given if the student described borrowing as trading or exchanging 1 of a denomination for 10 of another (e.g., 1 hundred for 10 tens). Such a description was possible in answering the questions under subheading 6. Fail was given if the student did not mention trading or exchanging.

Table 5

Distribution of Bug Types in Mapping and Prohibition Groups

Time of Test	Group	
	Mapping	Prohibition
Pre		
A *	0	0
B	6	4
C	2	3
D	0	1
E	1	1
Post-1		
A	2	3
B	3	3
C	3	1
D	1	0
E	0	2
Post-2		
A	2	3
B	3	2
C	4	2
D	0	0
E	0	2

* A means a correct algorithm, B-D are classes of bugs listed in Table 3.

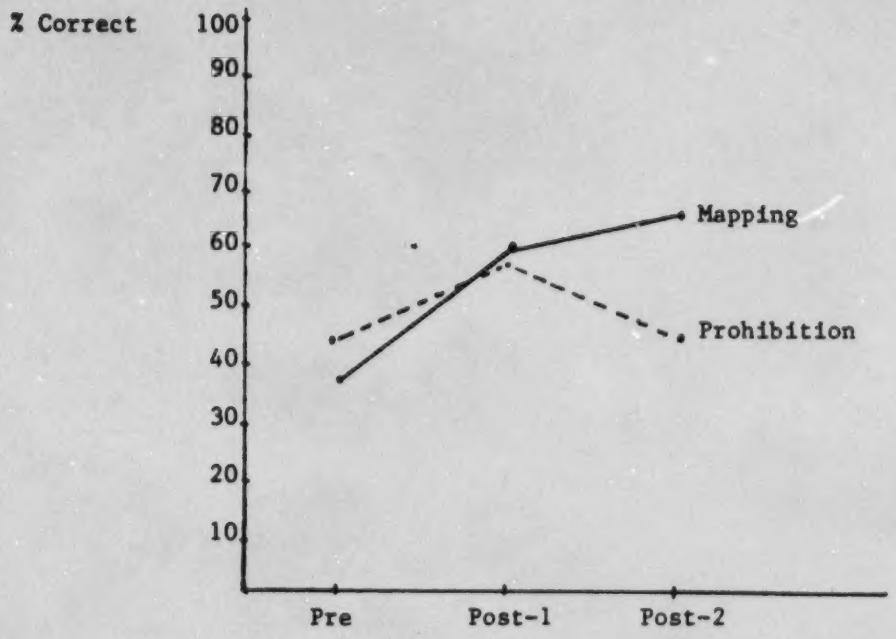


Figure 10. Percent correct on value questions.

each group honored the rule of compensation and no students in either group used the exchange metaphor to explain borrowing.

Pre-Post Comparison

Performance of Written Subtraction. Contrary to our expectations, mapping instruction was not effective, on the whole, in correcting children's bugs. At the immediate posttest (Post-1), only 2 out of 9 children in the mapping group performed borrowing without bugs (cf. Table 5). By the delayed posttest (Post-2), 1 of these 2 students had gone back to her original bug. However, another student had adopted a correct algorithm, so that there was a total of 2 students with no bugs at Post-2.

The prohibition instruction was not very successful as a cure for bugs, either, as only 3 out of 9 children had correct algorithms at Post-1 (cf. Table 5), and 2 out of the 3 went back to their bugs at Post-2. Two more children adopted correct algorithms at Post-2 but these improvements were most likely not due to the instruction.

One finding to note is that there were children in both groups who developed new bugs at Post-2 (4 who did so in mapping, and 3 in prohibition). We had expected new bugs to occur in the prohibition group, because that instruction focused on correcting specific bugs and did not teach any principles that would prevent new bugs from being tried later. However, we had not expected new bugs to appear after mapping instruction, because mapping was designed to teach understanding that would prevent all types of bugs.

As noted earlier in the description of the procedure, there was considerable variation in the length of time between Post-1 and Post-2, which raises the possibility that length of delay may be related to performance. Looking at the mapping and prohibition groups together, there was a modest relation between the length of delay and continued presence vs. absence of bugs, r=.38. A scatterplot diagram depicting the mapping and prohibition groups' scores separately (Fig. 11) showed that the basic pattern was the same in both groups. This correlation indicates that students who had a longer delay between Post-1 and Post-2 were more likely to have bugs in their written performance at Post-2 than those who had a shorter delay.

Understanding the semantics of written subtraction. As Figure 10 shows, the mapping group improved from Pre to Post-2 in their understanding of the values of the borrow digits, whereas the prohibition group did not. An ANOVA was performed on the value scores, comparing the mapping and prohibition groups at Pre, Post-1, and Post-2. There was no main effect for instruction, but the Instruction X Time-of-Test interaction was significant ($F(2,32) = 3.66$, $p<.05$). Newman-Keuls comparisons showed a significant difference between Pre and Post-2 scores for the mapping group ($p<.01$), but no such difference for the prohibition group. Similarly, in the mapping group there was an increase in the number of students who understood the rule of compensation (from 2 to 4) (cf. Figure 12), in the number who understood the goal structure (from 2 to 5) (cf. Figure 13). whereas there were no such increases in the prohibition group. Chi-square tests revealed a reliable difference between mapping and prohibition in number of students knowing the goal

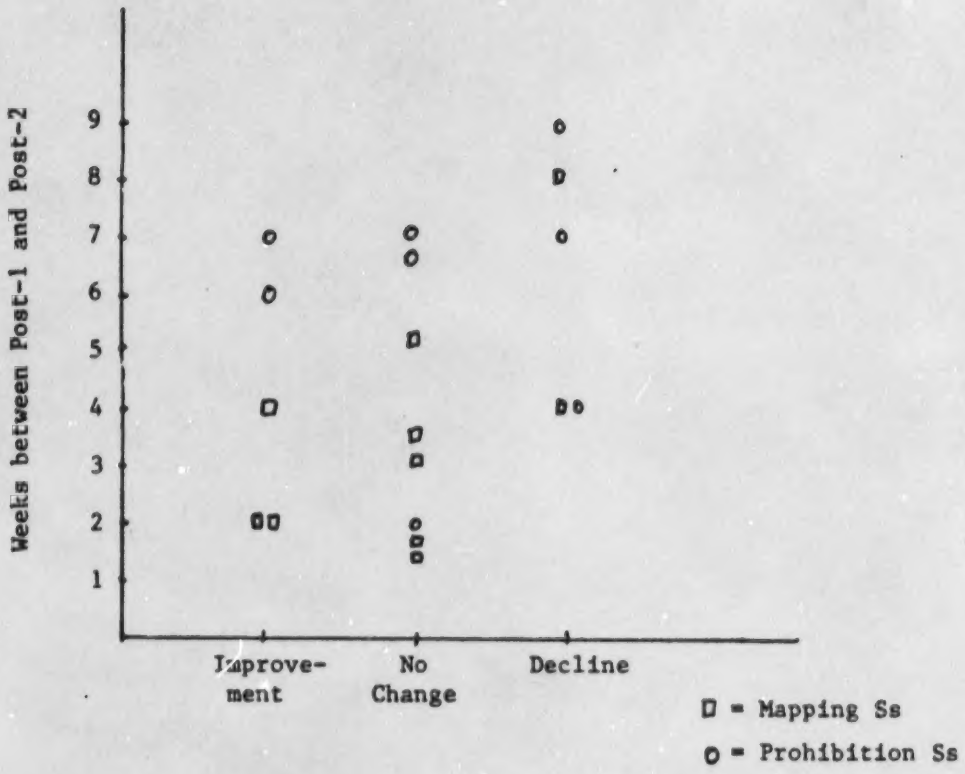


Figure 11. Length of delay between Post-1 and Post-2 and effect on performance on subtraction test.

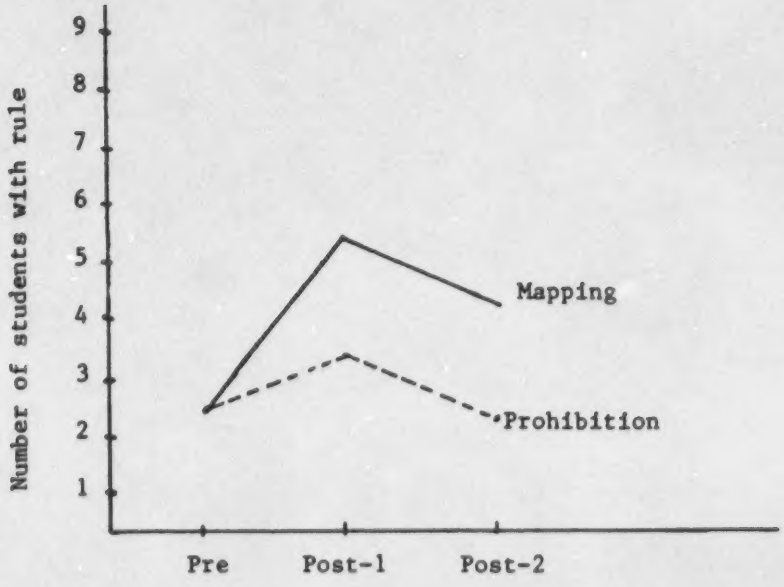


Figure 12. Rule of compensation.

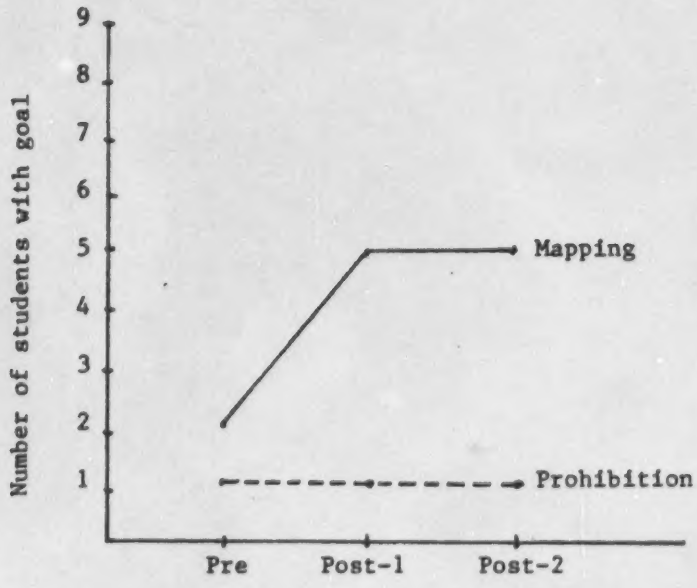


Figure 13. Top-level goal.

structure ($\chi^2 = 4.0$, $p < .05$), but not in the number of students who understood compensation Overall, then, the mapping group made significant gains in understanding the semantics of subtraction, and did better than the prohibition group, but they still fell short of complete understanding.

None of the students in either group used the exchange metaphor. However, one student in the mapping group invented another equally valid metaphor following instruction, which we will call the division-into-parts metaphor. It says that borrowing is a breaking up of 1 of denomination x into 10 parts, which are of the next lower denomination. So, according to this metaphor, borrowing from the hundreds column would mean taking 1 hundred and breaking it up into 10 tens, which are then placed into the tens column. Table 6 shows how our student used the division-into-parts idea to explain borrowing.

We will now examine the different patterns of understanding in individual children. What kinds of patterns would we expect to see? According to the Resnick et al. (1980) model of understanding (MOLLY), we would expect the four components of understanding (viz., goal structure, values, compensation, preserving the value of the top number) to be present together. These components are integrated in the MOLLY model by the controlling idea, acquired through mapping instruction, that borrowing is an exchange of 1x for 10y. In other words, once the MOLLY model conceptualizes borrowing as a 1-for-10 exchange, it also recognizes the correct values of borrow digits, that 1-for-10 exchanges satisfy the goal of getting more, that borrowing involves compensation between columns, and that it preserves the value of the top number.

Table 6

Example of the Division-into-Parts Metaphor

E: (shows S a problem that has been worked out: $\begin{array}{r} {}^{1}\not{2}\,{}^{9}\not{0}\,{}^{9}\not{0}15 \\ -\ 8\ 1\ 9 \\ \hline \end{array}$)

Here is another problem. Can you tell me where this number came from? (points to 9 above 0 in tens column)

S: This number came from the hundred's column.

E: Ok, and how does that work?

S: It came from the tens column, I mean. Cause right here, when you were in the thousands column, you had to cross out the 2 and make that one thousand. And then, one thousand in the tens column -- instead of taking one thousand, you break that into hundreds. And you take 10 hundred and then put 10 hundred there (in the hundreds column) and then you take out 1 hundred from the 10 hundred and put it into the tens column. But you broke it up and put it into tens. And you took 1 ten from the tens column and broke it up into ones and put it into the ones column.

As indicated above, none of the students gave evidence that they thought of borrowing as a 1-for-10 exchange. Thus, these students did not seem to acquire the central component of understanding in the MOLLY model. However, a number of them acquired some of the other components, such as the rule of compensation, and one of them, JAN, developed the division-into-parts metaphor, which is similar to exchange. Considering what this sample of students (pooling the mapping and prohibition groups) did know, as measured by the tests, we can see five patterns of understanding in the data (cf. Table 7). These patterns can be thought of as levels of understanding, because each pattern includes more understanding than the previous one. Three of the five scores on Table 4 were used to determine the levels of understanding: Score 3 (percent correct on values of borrow digits), Score 4 (0 or 1 for honoring the rule of compensation), and Score 5 (0 or 1 for mentioning either the exchange or division-into-parts metaphor).

At Level 1 (cf. Table 7), a child showed no evidence of knowing the value of borrow digits, compensation, or an exchange or division metaphor. There were two children at this level in the mapping group at the Pretest (cf. Table 8 and Figure 14).

There were more children at Level 2, which was the same as Level 1 except that at least one of the questions on values of borrow digits was answered correctly--either by stating the value or showing it with blocks.

Table 7

Five Levels of Understanding

	Defining Components			
Level	Identifying the correct value of at least 1 digit in borrowing	Using the rule of compensation for adjacent borrows	Identifying all values correctly	Using Exchange or Division into Parts Metaphor
1	0	0	0	0
2	1	0	0	0
3	1	1	0	0
4	1	1	1	0
5	1	1	1	1

1 = present
0 = absent

Table 9

Level of Understanding at Pretest and Delayed Posttest
in Mapping and Prohibition Groups

	Level of Understanding				
Assessment	1	2	3	4	5
Mapping					
Pretest	2	5	2	0	0
Post-2	0	5	2	1	1
Prohibition					
Pretest	0	7	2	0	0
Post-2	0	7	2	0	0

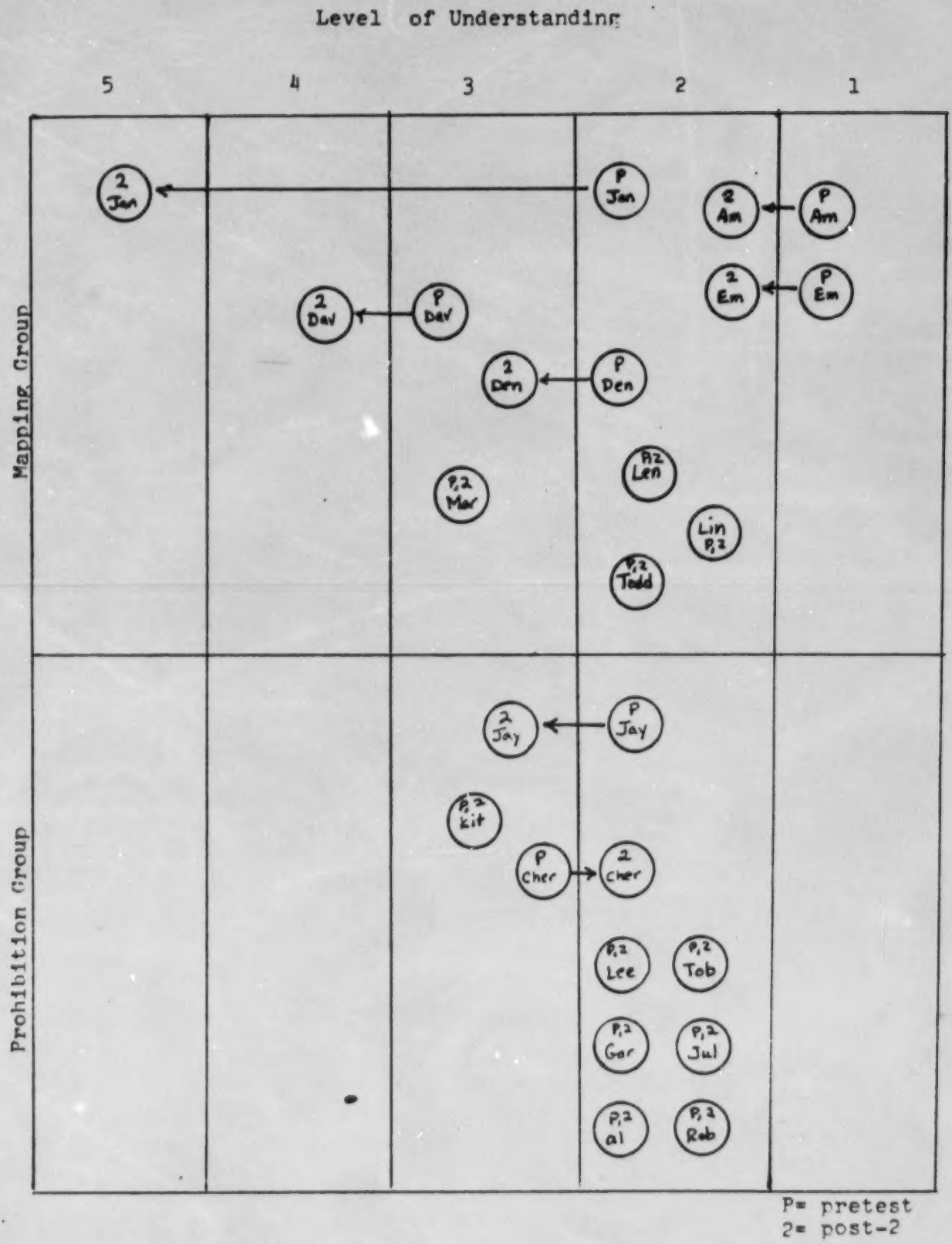


Figure 14. Changes in individuals' understanding.

At the next level, 3, the child was able to use the rule of compensation to identify borrowed quantities in the hundreds-to-tens borrow. In other words, he or she would say that in $\overset{3}{\not{4}}\,'2\ 8$, 100 was taken from the hundreds column and 100 was put into the tens column. Still, at this level, the child failed to identify some of the other values, e.g., the 9 in $\overset{4\ 9}{\not{5}\ \not{0}}\,'2$.

At Level 4, the child answered all questions about values correctly. Thus, the child honored the rule of compensation not only in hundreds-to-tens borrows, but in borrowing across zero as well. He or she also named the value of crossed out digits correctly.

Finally, at Level 5, the child answered all questions about values correctly and used a metaphor to explain borrowing.

The responses to the question about the goal structure (Score 2) were not included in defining the five levels of understanding because they did not fit into the hierarchy. (I.e., some students who knew the rule of compensation did not answer the goal question correctly, and on the other hand, some students who did not know the rule of compensation <u>did</u> answer the goal question correctly.) One reason why the response to this question did not fit into the hierarchy may be that the question was an inadequate measure of knowing the goal structure. The question, again (cf. Appendix D, pp. 4-5), asked the child whether or not it was possible to do the problem 547-824 and why (or why not). A correct response was to say, "No, because you can't take 824 from 547." A number of the students said that they could do the first two columns but not the leftmost, "because you can't take 8 from 5." we considered such an answer a failure to show understanding of the top-level goal to subtract the whole bottom number from the

whole top number. However, it is possible that the children who gave this answer did understand that goal, but just failed to verbalize it. Some evidence for this is that when the experimenter asked them again whether they could solve the whole problem, they said no. If we assumed that the children who gave the response just described did know the top-level goal, then we could say that, for the present group of children, knowing the goal structure fits into the hierarchy between Levels 2 and 3. In other words, some children at Level 2 knew the goal structure, and some did not, but all those at Levels 3,4, and 5 did know it.

Figure 14 shows that each student's performance in this study fit one of the five levels of understanding just defined. (Although the Post-1 levels are not shown in Figure 14, each student one of the five levels for that test, also.) Figure 14 also confirms that there was more improvement in the mapping group than in the prohibition group. Specifically, in the mapping group, 4 individuals advanced 1 level and 1 individual advanced 3 levels, whereas in the prohibition group, only 1 student advanced 1 level (cf. Table 9).

Examining the data in Figure 14 with the MOLLY model in mind, several points can be addressed. First, it appears that the understanding that is interconnected in the MOLLY model is not necessarily interconnected in children's minds--even after mapping instruction. For example, many of the children were at Level 2, which means that they could identify the values of some of the borrow digits correctly (e.g., saying that the 9 in $\overset{3}{\cancel{4}}\,\overset{9}{\cancel{0}}\,'2 = 90$) without demonstrating any of the other aspects of understanding (e.g., without recognizing that borrowing involves compensation). Second, the role

of the exchange metaphor in understanding borrowing is unclear. In the MOLLY model, the exchange metaphor was the means by which knowledge about the blocks was transferred to the written borrowing algorithm. And, once this knowledge was in place, the exchange metaphor (a schema) organized it for answering questions and for planning procedures. By contrast, in the present group of children, the exchange metaphor did not seem to play such a role in their understanding. In fact, none of the children even mentioned the idea of trading or exchanging when answering the questions about how much was borrowed. How, then, were these children able to answer many of these questions correctly? One possibility is that they simply had a rule for calculating the values of digits, such as the following: multiply the face value of a digit by its place value. For example, in the problem $\begin{array}{r} 4\ 9 \\ \not{5}\ \not{0}\ 3 \\ -\ 2\ 4 \end{array}$ they could use this rule to say that the 4 = 400, the 9 = 90, and the 13 = 13, without thinking about exchanges at all. Another possibility is that these children imagined how each digit would be represented in blocks (e.g., the 9 would be shown as 9 tens) and then named the blocks (e.g. "9 tens") as the answer. Either of these possibilities suggests that during mapping instruction the children may have been learning an expanded code map between columns and block values (that would include borrowed quantities as well as regular digits) rather than learning an operations map between trading and borrowing. In retrospect, such a result is not surprising, because the mapping instruction emphasized the task of making the blocks display match the written display. There was no attempt made by the experimenter to point out how trading and borrowing are alike--partly because we expected this to be obvious to the students.

Relationship between Understanding and Written Performance

Figures 15 and 16 depict each student's level of understanding and type of written algorithm at Pre, Post-1 and Post-2. These figures allow us to see the relationship between written performance and understanding. Figures 15 and 16 show that a high level of understanding (level 4 or 5) was associated with a correct written algorithm (class A) (e.g., Jan-2; Dav-2 in Figure 15). However, correct written performance was not strongly associated with a high level of understanding (e.g., Jay-1, Lee-2, Tob-2 in Figure 16). Looking at improvement in understanding and written performance, regardless of level, we see no clear relationships in either the mapping group or the prohibition group.

One interpretation of these results is that knowledge of the semantics does not facilitate written performance unless the knowledge is nearly complete--i.e., unless it includes knowledge of values of all digits and the rule of compensation. This interpretation makes sense in light of the hypothesis discussed earlier concerning how the semantics can be used to aid written performance. The hypothesis was that the semantics, in the form of a goal structure and constraints, can be used in planning the steps in the borrowing algorithm and in checking the legality of steps already performed. It is important to note that if the relevant constraint in borrowing (viz., that increments must be compensated for by equivalent decrements) is to be used in either planning or checking, the child who is using the constraint must know the values of individual digits involved in the compensation. For example, if the child is checking the borrowing in the problem in Figure 17 according to the compensation constraint, he

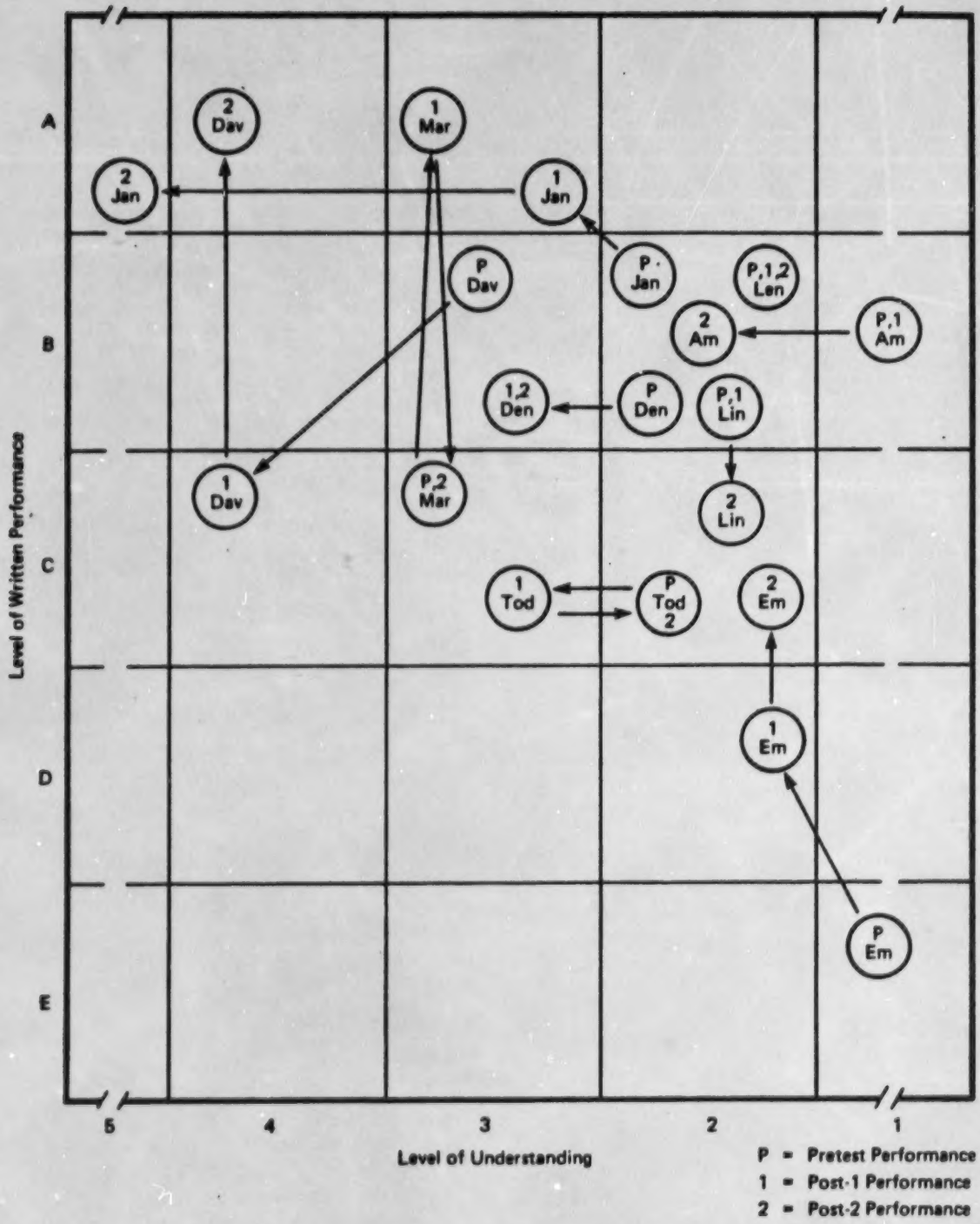


Figure 15. Individuals in the mapping group.

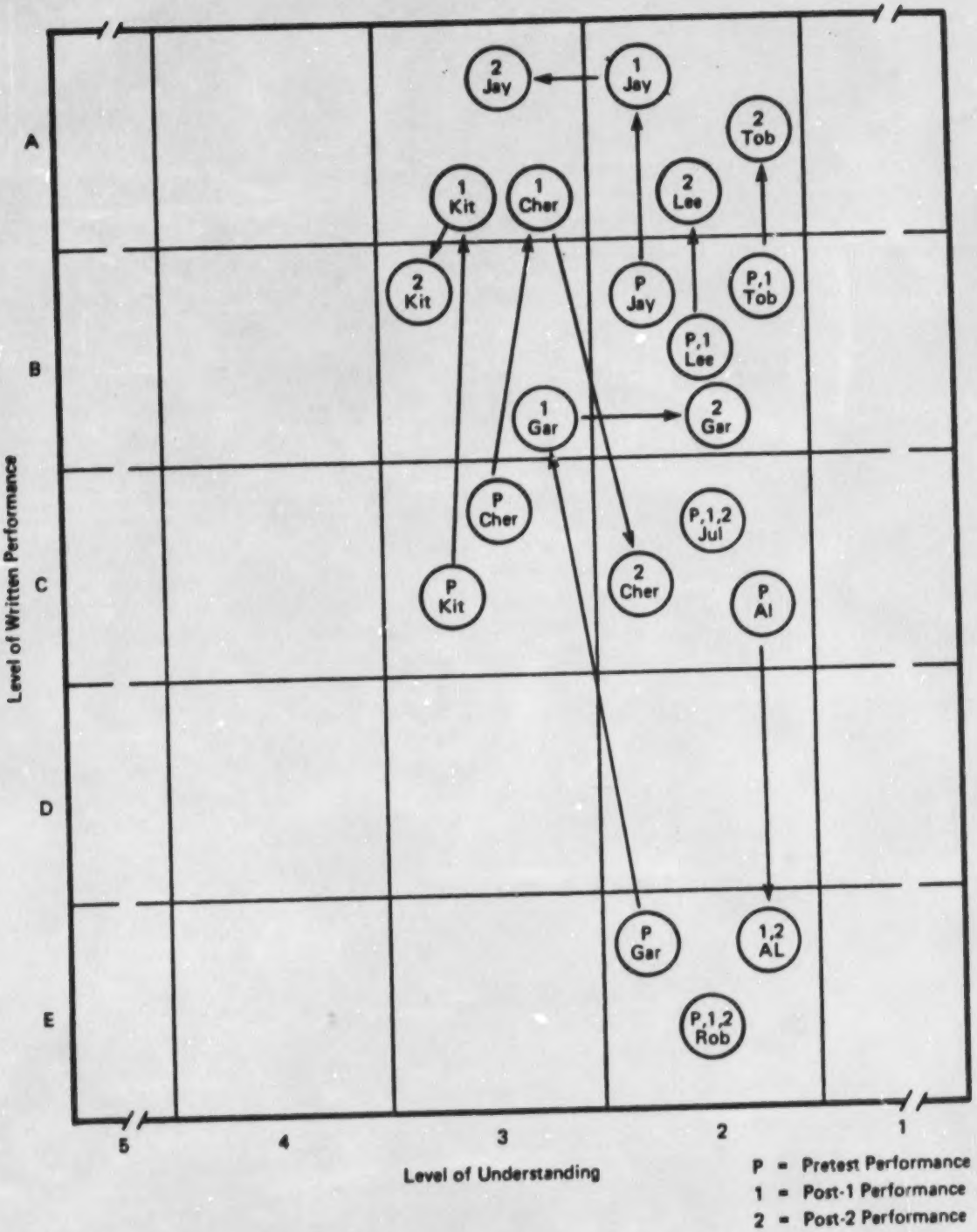


Figure 16. Individuals in the prohibition group.

or she must be able to recognize that the 1 written in the tens column is worth 100, whereas if it was written in the hundreds column it would be worth 1000.

$$\begin{array}{r} {\scriptstyle 2\ \ \ \ } \\ 3\,0\,{}'5\ 7 \\ -\ 1\ 8\ 3 \\ \hline 4 \end{array}$$

Figure 17.

Thus, given that knowing the values of digits is critical for using the semantics in written performance, it is not surprising that the students in this study who did not know all the values also did not use the rule of compensation to correct their bugs in written performance.

What Happened during Mapping Instruction

Of the nine children who were given mapping instruction only two (Janet and David) emerged from the instruction with complete or nearly complete knowledge of the semantics of written subtraction and with a correct written algorithm. Comparing their protocols with those of the other students, we wanted to see whether there were differences in the way they approached the mapping task that might explain why they seemed to learn more than the other students.

As described earlier (pp. 10-11) there are two features of mapping instruction that seem to be important to its effectiveness in teaching the semantics. The first is that the instruction leads the child to think about the blocks and do a first step in blocks before writing anything. Starting with the blocks, we argued, prompts the child to think about the subtraction problem in terms of goals and

constraints inherent in the blocks medium. The second important feature of the mapping instruction is that it forces the child to alternate continually between blocks moves and written moves, through the process of recording blocks moves in writing. The recording process is designed to help the child associate blocks values with written digits, see the analogy between trading and borrowing, and transfer the blocks semantics to the written digits and operations.

Because the students in this study had received pre-training in blocks subtraction, we expected them to be able to follow the mapping instruction with little difficulty. The pre-training had given them practice in working with the blocks goals and constraints, so that the only new part in mapping was to record each block move on the written problem. In spite of this, some of the students in the study had considerable difficulty with the mapping procedure. There were three features of students' protocols that we considered to be indicative of this difficulty: 1) doing several written steps before the appropriate blocks steps had been performed, 2) making errors in written steps that were consistent with the student's bug, and 3) being interrupted or prompted often by the experimenter for these or other reasons. At the beginning of the instruction, all of the students were interrupted many times by the experimenter, either to stop them from continuing to do written steps without the blocks or to guide them in how to record a blocks move. What distinguished Janet and David, who knew most of the semantics at Post-2, from all but one of the other students (Todd) was that by about the third subtraction problem, they did the mapping procedure with little interruption from the experimenter and they did not make any errors that could be traced

to their written bugs. Thus, Janet and David seem to have learned the mapping procedure more successfully than the others. Perhaps, then, some skill at doing the mapping procedure itself is necessary before one can think about how the blocks goals and constraints relate to written subtraction.

The other student, Todd, demonstrates that skill at the mapping procedure is not sufficient, however, for acquiring the semantics of subtraction. What other features of one's approach to mapping instruction may be important? Comparing Janet and David to Todd, we noticed that Janet and David verbalized the values of the digits more often during mapping. The verbalization may be important in strengthening the links in memory between digits and their values. Another difference between Todd and the other two students was that Janet and David were interrupted in the beginning because they did a series of correct written steps without doing blocks steps, whereas Todd was interrupted because he hesitated and made errors in trying to record the blocks moves. We think this difference indicated that Janet and David were thinking about their normal, previously established algorithms for written subtraction when they were asked to record the blocks moves, and that Todd was actually trying to record the blocks moves, forgetting about how he normally did written subtraction. It may be that Todd, who did not improve either in the semantics or in the written algorithm, failed to connect the mapping procedure with his usual method for doing written subtraction. In other words, he may have learned to record the blocks moves during mapping instruction as an extension of blocks subtraction, without realizing that the recordings correspond to his normal written

algorithm. In summary, these observations suggest that in order to learn the semantics of written subtraction through mapping instruction, a student may have to develop skill at the mapping procedure, verbalize the values of the digits frequently, and consciously associate his or her normal written algorithm with the recordings made as a part of the mapping process.

What Happened during Prohibition

For the majority of the children, little or no prohibition actually occurred. Six out of nine students did every problem correctly, including the problems chosen to elicit their bugs. Another child did every problem except one correctly; and in that problem, after two prohibitions, she chose the correct move herself. Thus, for these seven students, prohibition instruction actually consisted of rehearsal of the correct borrowing algorithm on a wide range of problems, including those chosen to elicit the students' bugs.

The remaining two students had great difficulty from start to finish, receiving about 6 prohibitions during almost every problem. Neither seemed to learn from the feedback, as they would make the same type of error repeatedly. One of the students had a Class E bug and the other showed a basic confusion about when to increment and when to subtract within a column. Thus, their difficulty in choosing correct moves after feedback may have been due to a real lack of knowledge of the correct algorithm. In any case, for these students, prohibition instruction consisted primarily of prohibitions--with little uninterrupted practice of the correct algorithm.

Conclusion

Mapping as a Cure for Bugs

In this study we found that neither the mapping instruction nor the prohibition instruction was successful in curing children's bugs. Because only a few students were cured of their bugs, it was not possible in this study to answer the question about the power of the prohibition aspect of mapping. Mapping instruction did produce gains in some aspects of understanding borrowing, but for most students, the increased understanding did not benefit their written performance--i.e., it did not prevent them from using buggy algorithms. One reason why increased understanding did not lead to better written performance may be that the increases in understanding were not comprehensive enough. It was suggested that mapping instruction might be more effective at teaching full understanding if it included more practice at verbalizing the values of the digits involved in borrowing, and more practice at the mapping procedure itself.

The Nature of Understanding Borrowing

This study also examined the nature of children's understanding of the borrowing algorithm. Understanding was defined as comprising four aspects, taken from the Resnick et al. (1980) computational model called MOLLY. These components were:

1. Knowing the goal structure of the borrowing algorithm.
2. Knowing the correct values of all the quantities involved in borrowing.
3. Recognizing a rule of compensation in borrowing, saying that the amount added to one column must be equal to the amount taken from another column.

a. Explaining compensation as a 1-10 exchange or a division into 10 parts.

b. Applying compensation to borrowing over zeros(s).

4. Recognizing that borrowing preserves the value of the whole top number.

On the basis of the MOLLY model we expected the children who received mapping instruction to acquire all four components of understanding as a result of recognizing the trade-borrow analogy during the instruction. In the MOLLY model, seeing this analogy enabled the system to understand that borrowing involves an exchange of 1 x for 10 y, and that this exchange maintains the value of the top number. What we found in the data was that the components of understanding, as measured be the pre and posttests, were not acquired together, and did not seem to be linked to the exchange metaphor. For example, a number of children were able to identify values of some of the borrow digits correctly following instruction, but never mentioned the idea of exchanging a ten for 10 ones, or a hundred for 10 tens. Such piecemeal acquisition of understanding, without a central organizing idea like exchange, suggested that the children were attending primarily to the code-map (i.e., between static block displays and numerals) during mapping instruction and either did not notice the operations map between borrowing and trading or did not use the operations map to interpret borrowing.

Summary

This study was designed to gather further evidence for the success of mapping instruction as a cure for bugs, so that the reasons for its success might be clarified. What the study has done instead is first to demonstrate that mapping instruction is not always successful in curing bugs, and that the relationship between bugs and understanding is not straightforward. For example, it is possible for a child to be able to explain where the 9 comes from in borrowing over zero, yet in the next few minutes, to fall back to a buggy algorithm. In addition, the study has shown that what we have called "understanding" is often acquired in pieces, rather than in the integrated structure built into the MOLLY model. And finally, it has raised some questions about what is learned in mapping instruction. In particular, the results suggested that in spite of the close procedural correspondence between trading and borrowing that is practiced in the instruction, children can miss the general trade-borrow analogy. A closer analysis of individual learning episodes may help clarify the factors that influence recognition of this analogy.

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Appendix

PreTest: Questions and Examples of Correct Responses

A. Prerequisite Knowledge

1. Value of regular digits and block denominations

a. Place value of digits

E: (shows card [444] First, I'd like you to show me this number using the blocks.

S: (puts out 4 hundreds blocks, 4 tens, 4 ones)

E: Good. Now show me which part of this pile stands for this number (points to the 4 in the tens column).

R1. S: (points to 4 tens)

E: How can you tell?

S: (They are both tens.)

E: What is the value of this number? (points to 4 in tens column)

R2. S: (40)

b. Biggest shape corresponds to leftmost digit

E: (puts 4 tens, 8 ones behind paper) There is a pile of blocks behind this paper. The biggest shape I have here is this (E shows S a ten). Guess which of these numbers is the amount that I have behind the paper? (shows S a card: [6 48 350 639]

R3. S: (48)

E: How can you tell?

R4. S: (It can't be 6, because that has no tens, and it can't be 350 or 639 because they have hundreds, which are bigger than tens)

E: Ok. (shows S the pile)

c. Constructing a partial display

E: (draws 3 blank spaces) I'm going to show you a number here, and I'd like you to show that number in blocks. (fills in spaces with 368) Can you show this number in blocks?

S: (puts out 3 hundreds, 6 tens, 8 ones)

E: (draws 3 more spaces) Now, I'm going to show you another number here, and I'd like you to show it in blocks. Only this time, I'm only going to show you part of the number first. (fills in hundreds space with 4, and ones space with 2). Can you show this much of the number in blocks?

R5. S: (puts out 4 hundreds, 2 ones)

E: (fills in tens space with 6) Now, which blocks are you going to add to show the whole number?

R6. S: (6 tens)

d. Comparison of column values

E: (writes 3 spaces in a row and 3 more underneath ---

I'm going to show you two numbers--one on top (points to top spaces) and one here (points to bottom spaces). (E fills in spaces as follows: 532

487)

Here are the two numbers. Which one is bigger?

S: (points to 532)

E: Now, I'm going to show you two more numbers. (writes 3 spaces in a row, and 3 underneath). This time, I'm only going to fill in 2 spaces at a time-- one in this number (points to top spaces) and one in this number (points to bottom spaces). Then, I'm going to ask you to guess which number is bigger-- this whole number, or this whole number. Which 2 spaces should I fill in first, so you can make a good guess?

87. S: (points to hundreds spaces)

E: (fills in spaces S chooses) (numbers are 374
623)

Now, do you want to make your guess now or do you think you need to see more spaces filled in first?

88. S: (The bottom number is bigger)

2. Trading

a. Outside of a subtraction problem

E: Show me 328 in blocks.

S: (gets out 3 hundreds, 2 tens, 8 ones)

E: (pushes S's block display behind paper) I am taking this block out (shows S a ten) and now I'm going to add these blocks back in (shows S 6 ones blocks). Will we still have 328 here?

89. S: (no)

S: (You took 10 out and only put 6 back in.)

E: What can I do to make it 328?

S: (add 4 more ones)

E: Go ahead and do it.

S: (adds 4 ones)

E: (shows S pile) How much do we have?

R10. S: (328)

b. Trading within a subtraction problem

(cf. item 4b)

3. Result mapping (i.e., match between blocks and written answers)

E: Now, would you do this problem in writing? 354 -228

S: $\overset{4}{3}\overset{}{8}14$ / -228 / 126

E: If you did this problem in blocks, what amount in blocks would you have left?

R11 S: (the same thing)

4. Blocks subtraction procedure - with trading

E: Now, go ahead and do this problem in the blocks. (354-228)

S: (sets up problem,

R12. trades 1 ten for 10 ones,

R13. subtracts 8 ones, 2 tens, 2 hundreds)

E: Did you get the same answer?

S: (yes)

B. Semantics of Subtraction

5. Goal of subtracting whole bottom number from whole top number.

E: (shows S card 547 / −324)

Another student was given this problem, but he didn't do the problem. Can you think of a reason why he didn't do it?

R14. S: (you can't subtract 824 from 547, because 824 is a bigger number.)

6. Verbal expression of values of borrow digits

a. Tens-to-ones borrow

E: Now I'm going to show you a problem that was done by another student. (shows card $\begin{array}{r} 6\overset{2}{\cancel{3}}5 \\ -217 \\ \hline 418 \end{array}$)

E: What was he doing when he crossed out that number?

R15. S: (He was borrowing.)

E: Where did he put what he borrowed?

S: (in the ones column)

E: How much did (you) put with this number? (points to ones digit)

R16. S: (10 ones)

[If S says "1"]

[E: How much is that 1 worth?

S: (10)

E: How much did he take from this number? (points to tens digit)

R17. S: (1 ten)

[If S says "1"]

[E: How much is that 1 worth?

S: (10)

b. Hundreds-to-tens borrow

E: (gets out card: $\begin{array}{r} \overset{3}{\cancel{4}}23 \\ -141 \\ \hline 282 \end{array}$)

Here is another problem that someone did. What was she doing when she crossed out this number? (4)

R18. S: (borrowing)

E: Where did she put what she borrowed?

S: (in the tens column)

E: How much did she put with this number? (2)

R19. S: (10 tens)

[If S says "10"]

[E: How much is that 10 worth?]

S: (100)

E: How much did she take from this number? (4)

R20. S: (1 hundred)

[If S says 1 or 10]

[E: How much is that ( ) worth?]

S: (100)

c. Borrow across 1 zero (hundreds-to-ones borrow)

E: (gets out next card - $\overset{4\,9}{\cancel{5}\,0\,3}$

$$\underline{-247}$$
$$256$$

In this problem, can you tell me where this number (points to the 9) came from?

R21. S: (It came from the hundreds column)

E: How did she get it from there?

S: (She borrowed it)

E: How much did she borrow?

R22. S: (100)

E: What happened to the ( ) that she borrowed?

(points to 4 above 5)

R 23. S: (It was exchanged for 10 tens, and 9 of them went to the tens column, and 1 was exchanged for 10 ones, which went to the ones column)

E: So how much is this number (points to 9) worth?

R 24. S: (90)

d. Borrow across 2 zeros (thousands-to-ones borrow)

E: (gets out next card $\begin{array}{r} {}^{1}{}^{9}{}^{9} \\ 2\,0\,0\,5 \\ -\ 8\,1\,9 \\ \hline 1\,1\,8\,6 \end{array}$

Here is another problem. (points to 9 in tens column)

Can you tell me where this number came from?

R 25. S: (It came from the thousands column)

E: How did he get that from there?

R 26. S: (First he borrowed from the thousands, and then he borrowed from the hundreds.)

E: How much did he borrow from here (points to thousands)?

R 27. S: (1000)

E: What happened to the ( ) that he borrowed? (points to 1 above 2)

R 28. S: (He exchanged the 1000 for 10 hundreds and put them in the hundreds column, then he took 1 of the hundreds, exchanged it for 10 tens, and put the tens in the tens column. Then, he took 1 of the tens, exchanged it for 10 ones, and put them into the ones column.)

E: And how much is this (9 in h) worth?

R 29. S: (900)

E: How much is this (9 in t) worth?

R30 6. S: (90)

7. Block expressions of values

a. Values of individual borrow digits

E: (gets out 536 - 247) Let's look at this problem. Can you show me how much this (points to the 1 next to the 6 in the ones column) is worth in blocks?

R31. S: (puts out 10 ones blocks)

E: Now, show me how much this (points to 4 above 5) is worth in blocks?

R32. S: (puts out 4 hundreds blocks)

E: OK. Now, can you show me how much this (points to 12 in tens column) is worth in blocks?

R33. S: (puts out 12 tens blocks)

b. Borrow-trade mapping

E: Shows S this written problem: 52 - 34. Let's say that you are going to do this problem in blocks. Here is the top number (sets up top number). Now, I'm going to do something in the writing and then I want you to do the same thing, only in the blocks. OK? (crosses out 5, writes 4 above it, writes a little 1 next to 2) See what I just did? Now, I want you to do the same thing with the blocks.

52
-34

4
~~5~~12
-34

R34. S: (trades a ten for 10 ones)

E: Shows S another written problem: 325-174. OK, let's set up this problem for doing it in the blocks. Here is the top number (gets 3Hs, 2Ts, 5ones) Now, I'm going to do something to this problem in

325
-174

writing, and I want you to do the same thing only in the blocks. (crosses out 3, writes 2 above, writes a little 1 in tens column next to 2). OK, what should you do in the blocks that is the same as what I did?

$\begin{array}{r} {\scriptstyle 2}\\ 3125\\ -\ 174\\ \hline \end{array}$

R35. S: (trades a hundred for 10 tens)

C. Borrowing Algorithm

7. Error checking

E: Now, let's say that you are the teacher. I'm going to show you how some students did subtraction - and you tell me if they are right or wrong. If something's wrong, tell me why you think so. (gives S sheet of paper with problems on it)

a. $\begin{array}{r} {\scriptstyle 3\ 4}\\ 402\\ -\ 37\\ \hline 535 \end{array}$

S: (It's wrong, because they forgot to write 9 above the 0, and you can't take 3 from 0)

b. $\begin{array}{r} {\scriptstyle 6}\\ 314\\ -107\\ \hline 207 \end{array}$

S: (It's right.)

c. $\begin{array}{r} {\scriptstyle 9}\\ 703\\ -149\\ \hline 654 \end{array}$

S: (It's wrong, because they forgot to borrow from the hundreds)

d. $\begin{array}{r} 536\\ -\ 28\\ \hline 512 \end{array}$

S: (It's wrong. They didn't borrow.)

e. 157

$$\begin{array}{r} -\ 84 \\ \hline 73 \end{array}$$

S: (It's right.)

f. 627

$$\begin{array}{r} -\ 4 \\ \hline 3 \end{array}$$

S: (It's wrong, because they did not write down the 6 and 2.)

8. Subtraction Test

Subtraction Test

Name _____ Grade _____

Teacher _____ Date _____

967 - 607	558 - 52	6078 - 4	74 - 28
477 - 2	30 - 19	108 - 60	353 - 148
1236 - 497	912 - 386	811 - 417	7261 - 5966
106 - 89	1712 - 417	8016 - 308	7002 - 3450
2005 - 73	604 - 206	5001 - 35	10013 - 716

Table D-1

Actual Order in which Pretest
was Presented to Children

Items
1. a
b
c
2.
5.
6.
7. a
3.
4.
7. b
8.
9.
1. d

Table D-2

Questions on Pretest about Values of Borrowed Quantities

	Tens-to-Ones Borrow		Hundreds-to-Tens Borrow			
Question	Amt. taken from Ts	Amt. added to ones	Amt. taken from Hs	Amt. added to Ts	Final value in Ts	Final value in Hs
Describe borrowing (naming values is optional)	R15	R15	R18	R18	R18	R18
Name values	R17	R16	R20	R19	--	--
Show values with blocks	R34	R34, R31	R35	R35	R33	R32

	Hundreds-tens-ones borrow (over zero)			Thous.-hundreds-tens-ones borrow (over zero)			
Question	Amt. taken from Hs	Amt. added to Ts	Amt. added to Ones	Amt. taken from Thous.	Amt. added to Hs	Amt. added to Ts	Amt. added to Ones
Describe borrowing	R21	R21, R23	R21, R23	R25	R25, R28	R25, R28	R25, R28
Name values	R22	R24	--	R27	R29	R30	--
Show values w. blocks	--	--	--	--	--	--	--